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New Solvable Lattice Models in Three Dimensions

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In this paper we establish a remarkable connection between two seemingly unrelated topics in the area of solvable lattice models. The first is the Zamolodchikov model, which is the only nontrivial model on a three-dimensional lattice so far solved. The second is the chiral Potts model on the square lattice and its generalization associated with the $U_q(sl(n))$ algebra, which is of current interest due to its connections with high-genus algebraic curves and with representations of quantum groups at roots of unity. We show that this last "$sl(n)$-generalized chiral Potts model" can be interpreted as a model on a three-dimensional simple cubic lattice consisting of n square-lattice layers with an N-valued ($N \geqslant 2$) spin at each site. Further, in the $N=2$ case this three-dimensional model reduces (after a modification of the boundary conditions) to the Zamolodchikov model we mentioned above.

KEY WORDS: Three-dimensional solvable models; Zamolodchikov model; generalized chiral Potts model; high-genus algebraic curves; quantum groups at roots of unity; commuting transfer matrices; Yang–Baxter equations; star-star relations.

INTRODUCTION

Despite the existence of numerous examples of an exact solvability or integrability of the two-dimensional models of field theory and statistical mechanics (see, e.g., refs. 1 and 2 for a review) there is very little that we can say so far about this phenomenon in three or more dimensions.

The only known solvable statistical model on a three-dimensional lattice which might not be reducible to a free field or Gaussian model (these latter are obviously solvable in any number of dimensions and are of limited interest) is the Zamolodchikov model.[(3)] This is an interaction-

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round-a-cube model with two-state spins on the sites of the simple cubic lattice. The Boltzmann weights of the model satisfy the tetrahedron equations,[3,4] which ensures the commutativity of the row-to-row transfer matrices and ultimately allows one to calculate the partition function per site in the thermodynamic limit.[5] Although this partition function was shown to be different[5] from that of its "weakly equivalent" three-dimensional free-fermion model,[6] the question about an exact relation between the Zamolodchikov model and the free-fermion model will remain unclear until the full spectrum of the transfer matrix is calculated, which has not yet been done. Note, in particular, that the two-layer Zamolodchikov model reduces[7] to the two-dimensional free-fermion model (or equivalently the checkerboard Ising model[8]).

In this paper we present a new solvable interaction-round-a-cube model on the cubic lattice with spins taking $N \geqslant 2$ distinct values. This can be regarded as a generalization of the Zamolodchikov model, reducing to it when $N=2$. We show that the transfer matrices of the model form two-parameter commuting families exactly as in the Zamolodchikov case. The rather unsatisfactory feature of the model is that the Boltzmann weights on the solvable manifold cannot be made real and nonnegative: in general they are complex. Nevertheless, we think that the model is interesting enough to be considered. It is possible that it is related to an interesting field theory. In particular, it is quite unlikely that for $N \geqslant 3$ it can be reduced to any conventional sort of the free system: even in the two-layer case it reduces to the two-dimensional chiral Potts model,[9–11] which seems to be the most complicated among the known solvable two-dimensional models. (For instance, in the scaling limit at the critical temperature it was conjectured[12,13] to be described by a Z_N-invariant conformal field theory with the central charge $c=2-6/N$).

The appearance of the chiral Potts model has a quite natural explanation described below, but still looks remarkable. Indeed, the same model has been already found in the center of various interesting connections. First, its Boltzmann weights do not have "the difference property"[9–11] and require high-genus algebraic functions for their parametrization.[14] Second, the chiral Potts model was shown to be a "descendant" of the six-vertex model[15] in the sense that its R-matrix satisfies the Yang–Baxter equation with two cyclic L-operators related to the R-matrix of the six-vertex model. Next, it turned out that the last connection can be extended by replacing the six-vertex model (with two states per edge) by an the n-state model[16] associated with the $U_q(sl(n))$ algebra. This resulted in the "$sl(n)$-generalized chiral Potts model,"[17] which is a two-dimensional model with spins that each take N^{n-1} values. Here we show that this last model [let us call it the $sl(n)$-CP model] can be interpreted as a model on a three-dimensional simple cubic lattice consisting of n square-lattice layers. At each site there is an N-valued spin. Further, this three-dimensional model is (to within a minor modification of the boundary conditions) the generalized Zamolodchikov model we mentioned above.

1. THE INTERACTION-ROUND-A-CUBE MODEL

Consider a simple cubic lattice $\mathscr{L}$ of M sites with periodic boundary conditions in each direction. At each site of $\mathscr{L}$ place a spin variable σ taking $N \geqslant 2$ distinct values $\sigma = 0, ..., N-1$, and allow all possible interactions of the spins within each elementary cube. The partition function reads

$$Z = \sum_{\text{spins}} \prod_{\text{cubes}} V(a \mid e, f, g \mid b, c, d \mid h) \tag{1.1}$$

where $a, ..., h$ are the eight spins of the cube arranged as in Fig. 1, and $V(a \mid e, f, g \mid b, c, d \mid h)$ is the Boltzmann weight of the spin configuration $a, ..., h$. The product is over all elementary cubes in $\mathscr{L}$.

Taking the lattice to have m horizontal layers and letting ϕ_i denote all the spins in layer i, one can rewrite (1.1) as

$$Z = \sum_{\phi_1} \sum_{\phi_2} \cdots \sum_{\phi_m} T_{\phi_1 \phi_2} T_{\phi_2 \phi_3} \cdots T_{\phi_m \phi_1} = \operatorname{Tr} T^m \tag{1.2}$$

where T is a layer-to-layer transfer matrix whose elements are the products of all the V functions of cubes between two adjacent layers. Clearly, T depends on the Boltzmann weight function V, so we can write it as $T(V)$.

As is known,[18,19] two transfer matrices $T(V)$ and $T(V')$ commute,

$$[T(V), T(V')] = 0 \tag{1.3}$$

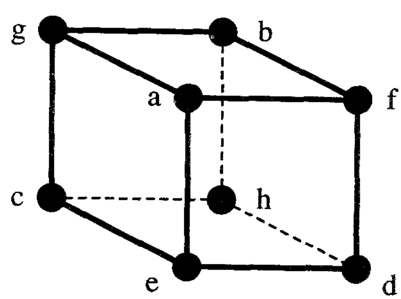


Fig. 1. Arrangement of the spins $a, ..., h$ on the corner sites of an elementary cube of the simple cubic lattice $\mathscr{L}$.

if there exist two other weight functions V'' and V''' such that V, V', V'', and V''' satisfy the tetrahedron relations[3,4] [see, e.g., Eq. (5.20) of ref. 5, which uses the same notations as here]. Alternatively one can require a slightly weaker (at least formally) and more transparent condition: the Yang–Baxter equation for composite "two-dimensional" weights. Indeed, consider a parallelepiped $\mathscr{P}$ formed by a line of n cubes in front-to-back direction with the periodic boundary (Fig. 2). Let $\alpha=(\alpha_1,..., \alpha_n)$, $\beta=(\beta_1,..., \beta_n)$, etc., denote the spins on the edges of $\mathscr{P}$. The Boltzmann weight of this parallelepiped reads

$$S(\alpha, \beta, \gamma, \delta)=\prod_{\text{cubes} \in \mathscr{P}} V(a,..., h) \tag{1.4}$$

Obviously $S(\alpha, \beta, \gamma, \delta)$ can be viewed as a Boltzmann weight function of some two-dimensional interaction-round-a-face model where each of the site spin variables takes N^n values. Hence, the two transfer matrices $T(V)$ and $T(V')$ will commute if there exists another weight function V'' such that S, S', and S'' satisfy the following Yang–Baxter equation:

$$\sum_\sigma S(\alpha, \beta, \gamma, \sigma)\, S'(\sigma, \gamma, \delta, \varepsilon)\, S''(\alpha, \sigma, \varepsilon, \kappa) = \sum_\sigma S(\kappa, \sigma, \delta, \varepsilon)\, S'(\alpha, \beta, \sigma, \kappa)\, S''(\beta, \gamma, \delta, \sigma) \tag{1.5}$$

where S' and S'' are given by (1.3) with V replaced by V' or V'', respectively.

There are simple transformations of V that do not change (1.1). For instance, if we multiply $V(a,..., h)$ by $F(a, f, b, g)/F(e, d, h, c)$, then each horizontal face of $\mathscr{L}$ acquires an F factor from the cube below it and a

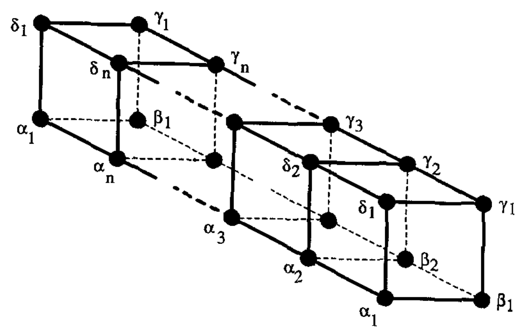


Fig. 2. A parallelepiped $\mathscr{P}$ formed by a line of n cubes in the front-to-back direction of the lattice with the periodic boundary.

canceling $(1/F)$ factor from the cube above. The effect on the transfer matrix T is to apply a diagonal similarity transformation. Provided F is the same for $T(V)$ and $T(V')$, the commutation relation is unaffected. This is an example of a "face-factor" transformation. Similarly, one could apply "edge-factor" and "site-factor" transformations that leave (1.1) and (1.3) unchanged. Note, however, that such transformations will in general destroy rotational symmetry of the weight function V (if any), so that symmetry properties of the partition function would be evident only in one specific "gauge."

2. THE BOLTZMANN WEIGHTS

Denote

$$\omega = \exp(2\pi i/N), \qquad \omega^{1/2} = \exp(\pi i/N) \tag{2.1}$$

Taking x to be a complex parameter and l to be an integer, $0 \leqslant l \leqslant N-1$, define a function $w(x, l)$ such that

$$\frac{w(x, l)}{w(x, 0)} = [\Delta(x)]^l \prod_{k=1}^{l} (1-\omega^k x)^{-1} \tag{2.2}$$

$$\Delta(x) = (1-x^N)^{1/N} \tag{2.3}$$

where $w(x, 0)$ is yet arbitrary. From these definitions it follows that

$$w(\omega^k x, 0)\, w(x, l+k) = w(x, k)\, w(\omega^k x, l) \tag{2.4}$$

$$w(x, l+N) = w(x, l) \tag{2.5}$$

Now, fix four complex parameters p, p', q, q', and define

$$\begin{aligned} s(k, l) &= \omega^{kl}, \qquad \Phi(k) = (\omega^{1/2})^{k(N+k)} \\ w_{pq}(k) &= w(p/q, k), \qquad \tilde{w}_{pq}(k) = w_{pq}(k)\, \Phi(k) \end{aligned} \tag{2.6}$$

Again, one can easily show that

$$s(a+N, b) = s(a, b+N) = s(a, b), \qquad \Phi(a+N) = \Phi(a) \tag{2.7}$$

$$s(a, b+c) = s(a, b)\, s(a, c) \tag{2.8}$$

With these notations define the weight function $V(a,..., h)$ as

$$V(a\,|\,efg\,|\,bcd\,|\,h) = \sum_{\sigma=0}^{N-1} v_\sigma(a\,|\,efg\,|\,bcd\,|\,h) \tag{2.9}$$

where

$$
\begin{aligned}
v_\sigma(a\,|\,efg\,|\,bcd\,|\,h)&\\
&= w_{p'p}(e-c-d+h)\, w_{p'p}^{-1}(a-g-f+b)\, s(c-h, d-h)\\
&\quad\times s(g, a-g-f+b)\\
&\quad\times \{w_{p'q}^{-1}(e-c-\sigma)\, w_{pq}(d-h-\sigma)\, w_{q'p}(\sigma-f+b)\\
&\quad\times \tilde{w}_{p'q'}(a-g-\sigma)\, s(\sigma, a-c-f+h)\}
\end{aligned}
\tag{2.10}
$$

where $w_{pq}^{-1}(l)$ denotes $1/w_{pq}(l)$. Clearly, V is a function of p, p', q, q' (as well as of the spins $a,..., h$), so that we can exhibit this dependence as

$$V = V(p, p', q, q') \tag{2.11}$$

(In fact, it depends only on three independent ratios among the variables p, p', q, q'). From (2.5), (2.7), and (2.9) it follows that V is unchanged upon independent increments of spins $a,..., h$ by the value of N, so we can regard them as defined modulo N. Substituting V in the form (2.9), (2.10) into (1.1), we see that Z is the partition function of an Ising-type model on a body-centered cubic lattice of $2M$ sites. A typical cube, with its center spin σ, is shown in Fig. 3. There are three-spin interactions on the shaded triangles, corresponding to the w-function inside the curly bracket in (2.10). In addition there are ω-type factors such as $s(a, \sigma)$ associated with the edges linking σ to a, f, c, h (these edges are denoted by heavy lines in Fig. 3). Note that four-spin interaction terms and ω-type factors before the curly brackets in (2.10) are just site-type and face-type equivalence transformation factors associated with the top and bottom faces. These cancel out of the partition function and are introduced in (2.10) merely for later

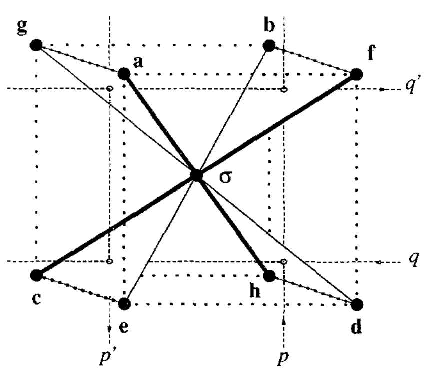


Fig. 3. A typical elementary cube of $\mathcal{L}$, with corner spins $a,..., h$ and the center spin σ.

convenience. One could also visualize the variables p, p', q, q' regarding them as rapidity-type variables associated with oriented dashed lines passing through the centers of the shaded triangles.

Now we want to prove the Yang–Baxter equation (1.5) for the "parallelepiped weights" (1.4). In fact we will do this for a slightly more general inhomogeneous model allowing the variables p, p', q, q' to vary from cube to cube in (1.4). Let p_i, p'_i, q_i, q'_i, $i=1,..., n$, denote the corresponding quadruplet of the variables for the ith cube in $\mathscr{P}$. These variables are not arbitrary: they should be constrained so that the quantity

$$\Gamma(p_i, p'_i, q_i, q'_i) = \text{independent of } i \tag{2.12}$$

is the same for any cube in $\mathscr{P}$, where

$$\Gamma(p, p', q, q') = -\frac{(p^N - q^N)(p'^N - q'^N)}{(p^N - q'^N)(p'^N - q^N)} \tag{2.12a}$$

Let us count the number of independent parameters on which the function S, (1.4), depends. Noting that the individual V function, (2.11), for the ith cube in $\mathscr{P}$ depends only on three ratios among the corresponding variables p_i, p'_i, q_i, q'_i, and taking into account the $n-1$ constraints (2.12), we are left with

$$3n - (n-1) = 2n + 1 \tag{2.13}$$

independent parameters. Further, for any two successive cubes i and $i+1$ define four numbers $k_{11}^{(i)}$, $k_{12}^{(i)}$, $k_{21}^{(i)}$, $k_{22}^{(i)}$, $i = 1,..., n-1$, satisfying the following four relations. The first one reads

$$p_{i+1}^N = \frac{k_{22}^{(i)} p_i^N + k_{21}^{(i)}}{k_{12}^{(i)} p_i^N + k_{11}^{(i)}}, \qquad i = 1,..., n-1 \tag{2.14}$$

and the other three are obtained from this by replacing the symbol p and p', q, or q', respectively. For any fixed i these four relations can be regarded as homogeneous linear system with k's as unknowns. Its determinant vanishes due to relations (2.12), so that a solution always exists. The normalization of the k's can be chosen such that

$$\det \mathbf{K}^{(i)} = 1, \qquad i = 1,..., n-1 \tag{2.15}$$

where $\mathbf{K}^{(i)}$ are two two-by-two matrices whose elements are $\mathbf{K}_{ab}^{(i)} = k_{ab}^{(i)}$, $a, b = 1, 2$.

Regarding the matrices $\mathbf{K}^{(i)}$, $i = 1,..., n-1$, as fixed, one could say that the equations (2.14) define an algebraic curve in the n-dimensional complex space $\mathbf{C}^n$, while the sequences $P=(p_1,..., p_n)$, $P'=(p'_1,..., p'_n)$, etc., specify points on this algebraic curve. The parameters $\{k_{ab}^{(i)}\}$ which specify the curve are called the "moduli" of the curve. Note that there are only $2n-3$ independent moduli relevant to our problem. In fact, multiplying each quadruplet of variables p_i, p'_i, q_i, q'_i by an overall factor λ_i $(i=1,..., n)$ leaves the weight function (1.4) unchanged, while the matrices $\mathbf{K}^{(i)}$ transform as

$$\mathbf{K}^{(i)} \to \Lambda_{i+1}^{-1} \mathbf{K}^{(i)} \Lambda_i \tag{2.16}$$

where $\Lambda_i = \mathrm{diag}(\lambda_i^{1/2}, \lambda_i^{-1/2})$. These transformations eliminate n degrees of freedom among the $3(n-1)$ independent matrix elements of the $\mathbf{K}^{(i)}$, satisfying Eqs. (2.15). The four points P, P', Q, Q' on the curve (2.14) (which will be referred to as rapidity variables) add four more degrees of freedom, which gives $(2n-3)+4=2n+1$ for the (continuous) parameter counting balance, exactly coinciding with (2.13) as it, of course, should be.

Thus, the weight function (1.4) is a function of the four rapidities P, P', Q, Q' and of the $2n-3$ moduli of the curve (2.14).[3] Assuming this latter dependence on moduli as implicit, and omitting the spin dependence, we can exhibit this as

$$S = S(P, P', Q, Q') \tag{2.17}$$

We also need to define a "modified model" similar to that considered in ref. 5 for the $N=2$ Zamolodchikov case. There are two sorts of vertical faces in $\mathscr{L}$: those whose perpendiculars run in front-to-back direction (such as *afde* and *gbhc* in Fig. 1), and those whose perpendiculars run right-to-left (*aecg* and *fdhb*). Call the former type FB, the latter RL. At the center of each FB face place a spin μ, with values $0,..., N-1$. Let the spins on *afde* and *gbhc* in Fig. 3 be μ and μ', respectively. Choose them so that

$$\sigma = \mu - \mu' \qquad (\mathrm{mod}\ N) \tag{2.18}$$

Do this for all cubes in $\mathscr{L}$. If σ' is the spin behind σ, and σ'' is the spin behind that, etc., then on using the cyclic boundary conditions we observe that

$$\sigma + \sigma' + \sigma'' + \cdots = (\mu - \mu') + (\mu' - \mu'') + (\mu'' - \mu''') + \cdots = 0 \qquad (\mathrm{mod}\ N) \tag{2.19}$$

[Each μ spin occurs twice with opposite signs. If $\mathscr{L}$ has n layers perpendicular to the front-to-back direction, then there are n σ-spins on the

[3] Note that it is *not* an algebraic function on the curve (2.14).

LHS of (2.19).] Thus we can use (2.18) only if the sum of each horizontal front-to-back line of σ-spins is constrained to be zero. We shall refer to the model subject to these constraints as the modified model. Substituting (2.9) in (1.4), one can split the resulting sum over the n σ-spins $\sigma_1,\ldots,\sigma_n$ (one for each cube in $\mathscr{P}$) into N terms

$$S = S_0 + \cdot + S_N \tag{2.20}$$

$$S_k = \sum_{\substack{\sigma_1,\ldots,\sigma_n \\ \sigma_1 + \cdots + \sigma_n = k}} \prod_{i=1}^{n} v_{\sigma_i}^{(i)} \tag{2.21}$$

Here we omitted all the argument dependences and denoted the function (2.10) for the ith cube as $v_\sigma^{(i)}$. Clearly, the term S_0 in the RHS of (2.20) is the "parallelepiped weight function" of the modified model.

A typical cube of $\mathscr{L}$ showing the new spins μ and μ' is drawn in Fig. 4. We see that we have now four-spin interactions on shaded rectangles. Because of the identity (2.8), there are ω-type factors associated with the eight edges (μ, c), (μ, h), (μ, a), (μ, f), (μ', c), (μ', h), (μ', a), (μ', f).

Now, we are ready to formulate an exact form of the Yang–Baxter equation (1.5). For convenience choose the number of vertical front-to-back layers of $\mathscr{L}$, n, to be relatively prime with N. Let P, P', Q, Q', R, R' be six points on the same curve (2.14); then the Yang–Baxter equation (1.5) is satisfied if we set S as in (2.17) and

$$S' = S(P, P', R, R'), \qquad S'' = S(Q, Q', R, R') \tag{2.22}$$

The same statement remains true if we replace S in (2.17), (2.22) by the parallelepiped weight function S_0 of the modified model. This latter state-

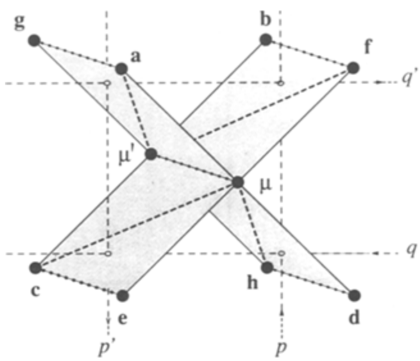


Fig. 4. The modified model obtained by changing from σ to μ spins according to (2.18). There are four-spin interactions on shaded rectangles; ω-type factors on edges are shown by solid or broken (but not dotted) lines.

ment related with the modified model will be proved in the next section. Here we assume that it is true and show that it implies the Yang–Baxter equation (1.5) with (2.17), (2.22) for the full weight function S.

To see this, one needs to use some simple symmetry properties of the weight function V which follow directly from its definition (2.9) and (2.10). In particular, one can easily show that the weight function S_0 defined in (2.21) remains unchanged upon the overall increment of all the spins on any edge of the parallelepiped $\mathscr{P}$. Remembering that the number of the front-to-back layers n is assumed to be relatively prime with N, one can conclude that S_0 depends only on the pairwise difference of the spins on the edges of $\mathscr{P}$. Setting $\langle\alpha\rangle = \alpha_1 + \cdots + \alpha_n$ and similarly for $\langle\beta\rangle$, $\langle\gamma\rangle$, and $\langle\delta\rangle$, taking into account that the spins on $\mathscr{P}$ obey the periodic boundary conditions and using the properties (2.4) of the function $w(x, l)$ entering the expression (2.6), one can easily show that

$$\begin{aligned} S_{\rho(k-l)}&(P, P', Q, Q' \mid \alpha, \beta, \gamma, \delta) \\ &= \omega^{(k-l)(\langle\alpha\rangle - \langle\beta\rangle + \langle\gamma\rangle - \langle\delta\rangle)} G_{kl}(P, P', Q, Q') \\ &\quad \times S_0(\omega^k P, \omega^k P', \omega^l Q, \omega^l Q' \mid \alpha, \beta, \gamma, \delta) \end{aligned} \tag{2.23}$$

where $\rho = -n \pmod{N}$ and $G_{kl}(P, P', Q, Q')$ is a scalar factor independent of the spins. This relation gives the change of $S_{\rho(k-l)}$ under overall increments of the spins, e.g., incrementing each of $\alpha_1, \ldots, \alpha_n$ by unity simply multiplies $S_{\rho(k-l)}$ by $\omega^{\rho(l-k)}$.

Substitute the sum (2.20) for each of the S functions in the Yang–Baxter equation (1.5) and expand the products. Then each side of these equations becomes a sum of N^3 terms. For example, the LHS of (1.5) will contain terms of the form

$$\begin{aligned} \sum_\sigma S_k&(P, P', Q, Q' \mid \alpha, \beta, \gamma, \sigma)\, S_l(P, P', R, R' \mid \sigma, \gamma, \delta, \varepsilon) \\ &\times S_m(Q, Q', R, R' \mid \alpha, \sigma, \beta, \varepsilon, \kappa), \qquad k, l, m = 0, \ldots, N-1 \end{aligned} \tag{2.24}$$

The terms with $k = l = m$ on both sides are equal: this is the Yang–Baxter equation for the modified model we just assumed. We shall now show that all the other pairs of the corresponding terms on both sides are equal as well, so that we in fact have N^3 distinct Yang–Baxter equations satisfied instead of only one. Labeling these Yang–Baxter equations by the three integers (k, l, m) introduced above, one can split them into two sets: the first set consists of those N^2 equations where $k = l - m \pmod{N}$ and the second set consists of the remaining $N^2(N-1)$ equations. All the Yang–Baxter equations in the first set are corollaries of the single equation with $k=l=m=0$. In fact, substituting the rapidity variables P, P', Q, Q', R, R' in the $(0,0,0)$ equation with P, P', $\omega^{-k}Q$, $\omega^{-k}Q'$, $\omega^{-l}R$, $\omega^{-l}R'$, respectively, and using (2.23), one obtains the $(\rho k, \rho l, \rho(l-k))$ equation. The equations in the second set are trivial in the sense that both sides vanish exactly. For example, substituting the expression (2.23) for the weight functions into (2.24) and performing the summation over the variables $\sigma_1,..., \sigma_n$ for any fixed set of their differences, one obtains a factor $\sum_j \omega^j$, causing the term to vanish.

Note that we considered the model with the front-to-back layer inhomogeneity only, implicitly assuming the translational invariance in the horizontal and vertical directions. The form of the rapidity dependence (2.17), (2.22) in the Yang–Baxter equation (1.5) suggests that one could introduce additional inhomogeneity in these directions as well, proceeding exactly as in two-dimensional case. Fixing the $2n-3$ independent moduli of the curve (2.14) to be the same for the whole lattice and taking it to have l column and m rows, introduce $2(l+m)$ rapidity variables P_j, P'_j, $j=1,..., l$; Q_j, Q'_j, $j=1,..., m$ (one pair for each column and row, respectively). Then assign the Boltzmann weight

$$S=S(P_j, P'_j, Q_k, Q'_k) \tag{2.25}$$

for the parallelepiped located at the intersection of the jth column and the kth row. The total number of the parameters defining this fully inhomogeneous model is equal to $2(l+m+n)-3$.

3. GENERALIZED CHIRAL POTTS MODEL

In this section we show that the modified three-dimensional model formulated in the previous section exactly coincides with the "generalized chiral Potts model" [or $sl(n)$-CP model] of ref. 17. Note that this latter model was originally formulated as a two-dimensional model, but here we give its three-dimensional interpretation.

First, recall the basic definitions of this model. Following ref. 17, consider an oriented two-dimensional square lattice $\mathscr{L}_{sq}$ and its medial lattice $\mathscr{L}'_{sq}$ (shown in Fig. 5 by solid and dashed lines, respectively). The oriented vertical (horizontal) lines of $\mathscr{L}'_{sq}$ carry rapidity variables P, P' (Q, Q') in alternating order (note that the orientations of rapidity lines shown by open arrows alternate, too). The edges of the lattice $\mathscr{L}_{sq}$ are oriented in such a way that all the SW–NE edges have the same (SW–NE) direction, while the SE–NW edges are oriented in a checkerboard order. (Note that our lattice $\mathscr{L}_{sq}$ is 90° anticlockwise rotated with respect to the lattice shown in Fig. 1 of ref. 17 with their rapidities p, p', q, q' replaced by Q, Q', P, P', respectively.)

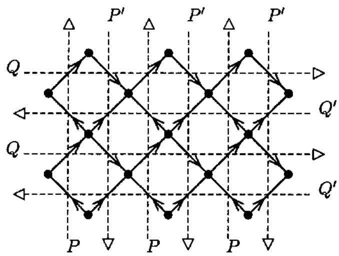


Fig. 5. An oriented square lattice $\mathscr{L}_{sq}$ and its medial lattice $\mathscr{L}'_{sq}$ shown by solid and dashed lines, respectively.

Each rapidity variable P here is represented by n 2-vectors $(h_i^+(P), h_i^-(P))$, $i=1,..., n$, $n \geqslant 2$, which specify a point of the algebraic curve defined by the relations

$$\begin{pmatrix} h_i^+(P)^N \\ h_i^-(P)^N \end{pmatrix} = \mathbf{K}^{(i)} \begin{pmatrix} h_{i+1}^+(P)^N \\ h_{i+1}^-(P)^N \end{pmatrix}, \qquad i=1,..., n-1 \tag{3.1}$$

where $\mathbf{K}^{(i)}$ are the same moduli matrices as defined after Eq. (2.15). Obviously this curve is related to our former curve (2.14), being its covering. In fact, setting

$$p_i = h_i^-(P)/h_i^+(P), \qquad i=1,..., n \tag{3.2}$$

one gets (2.14) from (3.1).

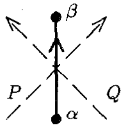
On each site of the lattice $\mathscr{L}_{sq}$ place n Z_N-spins, which are described by the local variable

$$\alpha = (\alpha_1,..., \alpha_n), \qquad \alpha_i = 0,..., N-1 \tag{3.3}$$

It is convenient to adopt a modulo n convention for the indices running through the values $1,..., n$, regarding them as belonging to Z_n. Denote

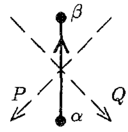
$$\begin{aligned} \hat{\alpha}_i &= (\alpha_i - \alpha_{i+1}) \pmod N, \qquad i \in Z_n \\ \langle \alpha \rangle &= \alpha_1 + \cdots + \alpha_n \\ \langle \hat{\alpha} \rangle &= \hat{\alpha}_1 + \cdots + \hat{\alpha}_{n-1} \end{aligned} \tag{3.4}$$

There are only two kinds of neighboring local state pairs, depending on the relative orientation of the dashed and solid lines as indicated in Figs. 6a and 6b, with states α and β, and Boltzmann weights $\bar{W}_{PQ}(\alpha, \beta)$ and $(\bar{W}_{PQ}(\alpha, \beta))^{-1}$ on the edges of $\mathscr{L}_{sq}$. The arrow from α to βs indicates that the argument is (α, β) rather than (β, α). The partition function of the



$= \overline{W}_{PQ}(\alpha,\beta)$

(a)



$= \dfrac{1}{\overline{W}_{QP}(\alpha,\beta)}$

(b)

Fig. 6. The two Boltzmann weights depending on the orientation of the spin pair with respect to rapidity lines P and Q.

model is defined as a sum over all possible configurations of spin variables of the products of the Boltzmann weights for all the edges of $\mathscr{L}_{\text{sq}}$.

To write down $\bar{W}_{PQ}(\alpha, \beta)$, introduce the function $g_{PQ}(\alpha, \beta)$, which possesses the properties

$$g_{PQ}(\alpha, \alpha) = g_{PQ}(\alpha, \beta)\, g_{PQ}(\beta, \gamma)\, g_{PQ}(\gamma, \alpha) = 1, \qquad \forall \alpha, \beta, \gamma \tag{3.5}$$

Then it is unambiguously defined by

$$g_{PQ}(\alpha, \alpha + \delta_i) = \frac{h_{i-1}^{+}(P)\, h_{i-1}^{-}(Q) - h_{i-1}^{-}(P)\, h_{i-1}^{+}(Q)\, \omega^{\hat{\alpha}_{i-1}}}{h_i^{+}(P)\, h_i^{-}(Q) - h_i^{-}(P)\, h_i^{+}(Q)\, \omega^{1+\hat{\alpha}_i}} \tag{3.6}$$

The symbol δ_i means a unit vector in the ith direction, i.e., all its components vanish except the value 1 in the ith place; $\omega = \exp(2\pi i/N)$.

The Boltzmann weight $\bar{W}_{PQ}(\alpha, \beta)$ has the form

$$\frac{\bar{W}_{PQ}(\alpha, \beta)}{\bar{W}_{PQ}(0, 0)} = \omega^{Q(\alpha, \beta)} g_{PQ}(0, \alpha - \beta) \tag{3.7}$$

where

$$Q(\alpha, \beta) = \sum_{i \in Z_n} \hat{\beta}_{i-1}(\alpha_i - \beta_i) \tag{3.8}$$

Iterating Eqs. (3.6), one can show that[20]

$$g_{PQ}(0, \alpha) = \frac{\prod_{k=1}^{\langle \hat{\alpha} \rangle} A_{PQ}^{(n)}(-k+1)}{\prod_{i=1}^{n-1} \prod_{k_i=1}^{\hat{\alpha}_i} A_{PQ}^{(i)}(k_i)} \tag{3.9}$$

where

$$A_{PQ}^{(i)}(k) = h_i^{+}(P)\, h_i^{-}(Q) - h_i^{-}(P)\, h_i^{+}(Q)\, \omega^k \tag{3.10}$$

Next, owing to the relations (3.1), the quantity

$$\Delta_{PQ}^{N} = \sum_{k=0}^{N-1} A_{PQ}^{(i)}(k) \tag{3.11}$$

is independent of i. Using this fact and the simple relation $\hat{\alpha}_n = -\langle\hat{\alpha}\rangle$ (mod N), we can further rewrite (3.9) as

$$g_{PQ}(0, \alpha) = \prod_{i=1}^{n} \prod_{k_i=1}^{\hat{\alpha}_i} \left\{ \frac{\Delta_{PQ}}{\Delta_{PQ}^{(i)}(k_i)} \right\} \tag{3.12}$$

Substituting now (3.12) into (3.7) and using the relations (3.2), we finally obtain

$$\frac{\bar{W}_{PQ}(\alpha, \beta)}{\bar{W}_{PQ}(0, 0)} = \prod_{i=1}^{n} \{\omega^{(\beta_i - \beta_{i+1})(\alpha_{i+1} - \beta_{i+1})} w(p_i/q_i, \alpha_i - \alpha_{i+1} - \beta_i + \beta_{i+1})\} \tag{3.13}$$

where $w(x, k)$ was defined in (2.2).

Note that there is some redundancy in the description of local spin varialbes: the spin Boltzmann weights (3.7) are unaffected by incrementing each of $\alpha_1, ..., \alpha_n$ by unity for any local state α on $\mathscr{L}_{sq}$. Thus, there are only $n-1$ relevant spin degrees of freedom at each site, say $n-1$ independent pairwise differences among $\alpha_1, ..., \alpha_n$. For later notational convenience we will not try to remove this redundancy here.

When the moduli matrices $\mathbf{K}^{(i)}$ are in general position, then the genus of the algebraic curve (3.1) is[17]

$$g = N^{2(n-1)}[(n-1)N - n] + 1$$

However, when all these matrices are equal to the identity matrix the genus is zero. As usual, one expects in the latter case the model to be critical. In fact, it was argued in ref. 20 that the scaling limit of the model in this case is described by an $sl(n)$-parafermion conformal field theory.[21, 22] In particular, when $n=2$ (two-layer case) the model reduces to the two-dimensional Z_N-invariant lattice model[13] whose scaling behavior is described by a Z_N-invariant conformal field theory of ref. 12.

One can group the sites of the lattice either into elementary stars or into elementary boxes as shown in Figs. 7 and 8, respectively. Denoting the Boltzmann weight of the star as

$$W_{\text{star}}(P, P', Q, Q' \mid \alpha, \beta, \gamma, \delta) = \sum_{\mu} \frac{\bar{W}_{PQ}(\beta, \mu)\, W1_{Q',P}(\mu, \gamma)\, \bar{W}_{P'Q'}(\delta, \mu)}{\bar{W}_{P'Q}(\alpha, \mu)} \tag{3.14}$$

and the Boltzmann weight of the box as

$$R(P, P', Q, Q' \mid \alpha, \beta, \gamma, \delta) = \frac{\bar{W}_{PQ}(\alpha, \delta)\, \bar{W}_{Q',P}(\delta, \gamma)\, \bar{W}_{P'Q'}(\gamma, \beta)}{\bar{W}_{P'Q}(\alpha, \beta)} \tag{3.15}$$

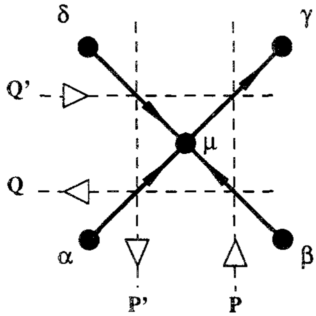


Fig. 7. An elementary star of $\mathscr{L}_{sq}$.

one can view the model either as an interaction-round-a-face with the face weight function given by (3.14) or as a vertex model with the vertex weight function given by (3.15). The latter weight function, R, will be called the R-matrix of the model. Clearly, the above two representations of the model are equivalent to each other (provided the periodic boundary conditions in the horizontal direction are assumed). In particular, they lead to the same row-to-row transfer matrix T_{sq}. Indeed, arranging the spin states as shown in Fig. 9, one can write the matrix elements of T_{sq} in the following two forms:

$$\begin{aligned}(T_{sq}(P, P', Q, Q'))_{\phi\phi'} &= \sum_{\{\mu\}} \prod_{i+1}^{M} R(P, P', Q, Q' \mid \phi_i, \mu_{i+1}, \phi'_i, \mu_i) \\ &= \prod_{i+1}^{M} W_{\text{star}}(P, P', Q, Q' \mid \phi_i, \phi_{i+1}, \phi'_{i+1}, \phi'_i) \qquad (3.16)\end{aligned}$$

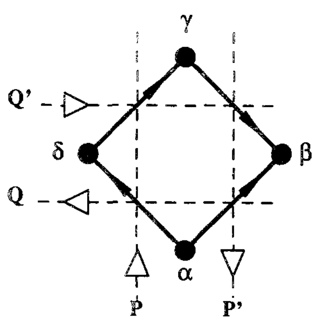


Fig. 8. An elementary box of $\mathscr{L}_{sq}$.

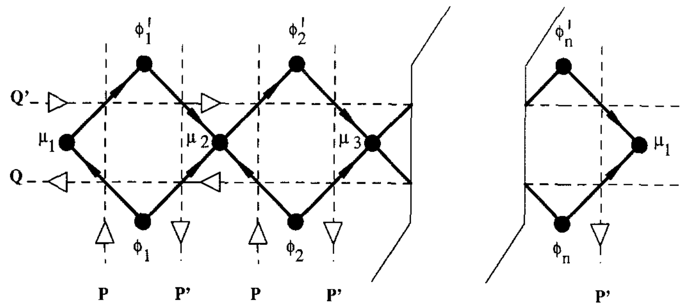


Fig. 9. Graphical representation of elements of the transfer matrix $(T_{\mathrm{sq}}(P, P', Q, Q'))_{\phi\phi'}$ as given by Eqs. (3.16).

Next, the R-matrix (3.15) satisfies the following Yang–Baxter equation[4]:

$$\sum_{\mu\nu\lambda} R(P, P', Q, Q' \mid \alpha\beta\nu\mu)\, R(P, P', R, R' \mid \nu\gamma\delta\lambda)\, R(Q, Q', R, R' \mid \mu\lambda\varepsilon\kappa)$$
$$= \sum_{\mu\nu\lambda} R(Q, Q', R, R' \mid \beta\gamma\nu\mu)\, R(P, P', R, R' \mid \alpha\mu\lambda\kappa)\, R(P, P', Q, Q' \mid \lambda\nu\delta\varepsilon) \qquad (3.17)$$

which was originally conjectured in ref. 17 and then proved in ref. 23 for odd values of N and later in ref. 20 for any value of $N \geqslant 2$. Obviously there should exist a counterpart of this statement for the interaction-round-a-face formulation of the model. Before showing this, we establish another and, in fact, a more fundamental relation between the Boltzmann weights of the model. This is so-called "star–star" relation.[5] Setting $R' = Q' = P'$ in (3.17), using the property

$$\overline{W}_{PP}(\alpha, \beta) = \overline{W}_{PP}(0, 0) \prod_{i=1}^{n-1} \delta_{\alpha_i, \beta_i} \qquad (3.18)$$

and redenoting the rapidity variables, one can reduce this equation to the following form:

$$W^{(1)}_{\mathrm{star}}(P, P', Q, Q' \mid \alpha, \beta, \gamma, \delta) = W^{(2)}_{\mathrm{star}}(P, P', Q, Q' \mid \alpha, \beta, \gamma, \delta) \qquad (3.19)$$

[4] This is Eq. (0.15) of ref. 17 with their $S^{\gamma\delta}_{\alpha\beta}(P, P', Q, Q')$ replaced by our $R(P, P', Q, Q' \mid \alpha, \beta, \gamma, \delta)$.

[5] The authors are indebted to Dr. R. M. Kashaev for an explanation of the derivation of this relation given here.

where

$$W^{(1)}_{\rm star}(P,P',Q,Q'\,|\,\alpha,\beta,\gamma,\delta) = \bar{W}_{P'P}(\alpha,\beta)\,\bar{W}^{-1}_{P'P}(\delta,\gamma)\,W_{\rm star}(P,P',Q,Q'\,|\,\alpha,\beta,\gamma,\delta) \tag{3.20}$$

with $W_{\rm star}$ given in (3.14) and

$$W^{(2)}_{\rm star}(P,P',Q,Q'\,|\,\alpha,\beta,\gamma,\delta) = \bar{W}^{-1}_{Q'Q}(\alpha,\delta)\,\bar{W}_{Q'Q}(\beta,\gamma)\sum_{\mu}\frac{\bar{W}_{PQ}(\mu,\delta)\,\bar{W}_{Q',P}(\alpha,\mu)\,\bar{W}_{P'Q'}(\mu,\beta)}{\bar{W}_{P'Q}(\mu,\gamma)} \tag{3.21}$$

Equation (3.19) is a statement of equality of the Boltzmann weights of the two "stars" shown graphically in Fig. 10, so it can be naturally called the star–star relation. In the presented derivation it is the consequence of (3.17) and (3.18). Conversely, the star–star relation (3.19) implies the Yang–Baxter equation (3.17). The proof consists in a repeated application of (3.19) to the LHS of (3.17) so as to transform it to the RHS. (This process can be described graphically as moving one pair of rapidity lines, say P, P', through the intersection points of the other two pairs of rapidity lines). Similarly, one can prove the required Yang–Baxter equation for the interaction-round-a-face weight function

$$\begin{aligned}\sum_{\sigma} & W^{(1)}_{\rm star}(P,P',Q,Q'\,|\,\alpha,\beta,\gamma,\sigma)\,W^{(1)}_{\rm star}(P,P',R,R'\,|\,\sigma,\gamma,\delta,\varepsilon) \\ &\times W^{(1)}_{\rm star}(Q,Q',R,R'\,|\,\alpha,\sigma,\varepsilon,\kappa) \\ &= \sum_{\sigma} W^{(1)}_{\rm star}(Q,Q',R,R'\,|\,\beta,\gamma,\delta,\sigma)\,W^{(1)}_{\rm star}(P,P',R,R'\,|\,\alpha,\beta,\sigma,\kappa) \\ &\times W^{(1)}_{\rm star}(P,P',Q,Q'\,|\,\kappa,\sigma,\delta,\varepsilon)\end{aligned} \tag{3.22}$$

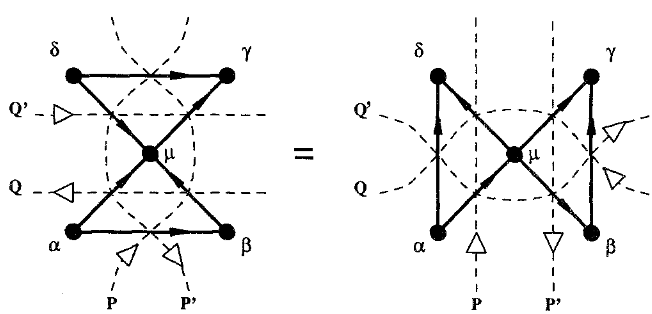


Fig. 10. The Boltzmann weights of these two stars are equal in virtue of the star–star relation (3.19).

Thus, both Yang–Baxter equations (3.17) and (3.22) are corollaries of the star–star relation (3.19). Note that for $n=2$ the model under consideration reduces (see Section 5) to the original chiral Potts model[11] and the star–star relation (3.19) is just a consequence of the star–triangle relation [Eq. (1) of ref. 11]. However, for $n \geqslant 3$ the corresponding star–triangle relation apparently does not exist (at least it is not known to the authors) and the star–star relation (3.19) seems to be the simplest relation of this type.

Note that the Yang–Baxter equation (3.22) has the same spin arrangement and the same rapidity dependence structure as in the Yang–Baxter equation (1.5) with (2.17), (2.22). In fact, these equations are more than just similar: we shall now show that the star weight function (3.20) exactly coincides with the parallelepiped weight function S_0, (2.21), of the "modified model" of Section 2,

$$W^{(1)}_{\text{star}}(P, P'Q, Q' \,|\, \alpha, \beta, \gamma, \delta) = S_0(P, P', Q, Q' \,|\, \alpha, \beta, \gamma, \delta) \tag{3.23}$$

completing thereby the proof of the commutativity of the transfer matrices for the three-dimensional model we consider in this paper.

The weight function $\bar{W}_{PQ}(\alpha, \beta)$ is associated with the edge of the two-dimensional square lattice connecting the sites with the states $\alpha = (\alpha_1,..., \alpha_n)$ and $\beta = (\beta_1,..., \beta_n)$ on them as shown in Fig. 6. Let us extend this edge in a third additional dimension perpendicular to the plane of the two-dimensional lattice to form a rectangle consisting of n squares, and place the spins $\alpha_1,..., \alpha_n$, etc., on the edges of this rectangle, as shown in Fig. 11. We also assume cyclic boundary conditions in the new dimension, considering the spins α_1, β_1 as next to α_n, β_n, respectively. Do this for all edges of $\mathscr{L}_{\text{sq}}$. Then it becomes the three-dimensional cubic lattice with an N-valued spin at each site. The weight function $\bar{W}_{PQ}(\alpha, \beta)$ is now regarded as a Boltzmann weight of the rectangle of Fig. 11. Remarkably, this weight function possesses a specific factorization property: the ith term in the product (3.13) depends only on the four spins α_i, β_i, β_{i+1}, α_{i+1} located at the corner of the ith elementary square within the rectangle, so that one can naturally regard it as a *local* Boltzmann weight of that square. This property allows us to interpret the model as a three-dimensional model with local interactions.

Substitute the expression (3.13) into the RHS of (3.20) and use the relation

$$\omega^{Q(\mu,\delta)+Q(\delta,\mu)} = \prod_{i=1}^{n} \Phi(\mu_i - \delta_i) \tag{3.24}$$

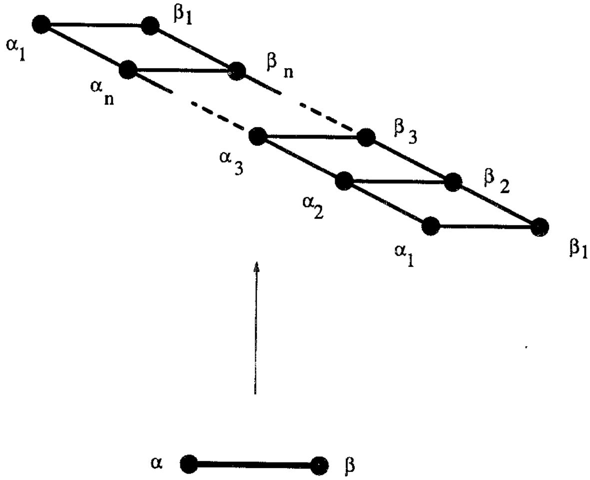


Fig. 11. Spin arrangement on the rectangular formed by the extension of a typical edge of $\mathscr{L}_{sq}$ in the third additional dimension perpendicular to the plane of the two-dimensional lattice.

where $Q(\alpha, \beta)$ and $\Phi(k)$ are defined in (3.8) and (2.6), respectively. Then, after canceling a number of ω-type factors, one can rewrite the RHS of (3.20) as

$$\sum_{\mu} \prod_{i=1}^{n} v^{(i)}_{(\mu_i - \mu_{i+1})}(\delta_i | \alpha_i, \gamma_i, \delta_{i+1} | \gamma_{i+1}, \alpha_{i+1}, \beta_i | \beta_{i+1}) \tag{3.25}$$

where $v^{(i)}_{\sigma}$ is defined by (2.10) with the parameters p, p', q, q' replaced by p_i, p'_i, q_i, q'_i respectively. This is the expression (2.21) for the weight function S_0.

Thus, we have proved the commutativity of the transfer matrices for the three-dimensional model with Boltzmann weight function (2.9). More precisely, we have proved this for the inhomogeneous model where the parameters p, p', q, q' can vary from one vertical front-to-back layer to the other [subject to the constraints (2.12)]. Let us recall the sequence of our arguments (in reverse order). The Yang–Baxter equation (3.17) for the R-matrix of the $sl(n)$-CP model (proved in refs. 20 and 23) implies the Yang–Baxter equation (3.22) with (3.23) for the parallelepiped weight function S_0 of the modified model, which in its turn leads to Eqs. (1.5), (2.17), and (2.22) and, hence, to the commutativity (1.3). The imposed restriction that the number of the vertical front-to-back layers n should be relatively prime with N is probably not essential. In fact, one could expect an existence of local integrability conditions, like the tetrahedron relations, in the model, which involve few different weight functions (2.9), rather than "global" constructions like (1.4). For instance, as we shall see in the next section, our three-dimensional model reduces to the Zamolodchikov model[3] when $N=2$, where the tetrahedron relations are known.[3, 4] [Note that these relations imply[18, 19] the Yang–Baxter equation (1.5) for an arbitrary number of layers n.] Alternatively, one could try to establish the three-dimensional star–star relations, which are also the local integrability conditions, but they are simpler than the tetrahedron relations. These star–star relations are known for the Zamolodchikov case and apparently could be generalized for a general value of N. A conjectured form of these relations is given below [Eq. (6.1) of Section 6].

4. THE ZAMOLODCHIKOV THREE-DIMENSIONAL MODEL

Following ref. 5, we consider the Zamolodchikov model in an interaction-round-a-cube formulation rather than in the original face-spin formulation of ref. 3. Then it is precisely a model of the type defined in Section 1 with spins taking two values, $N=2$.

In ref. 5 it was shown that up to face-type and site-type equivalence transformations discussed at the end of Section 1, the Boltzmann weight of the spin configuration $a,..., h$ around the elementary cube in Fig. 1 can be written in the form

$$V(a,..., h)=\sum_{\sigma=0}^{1} (-1)^{\sigma(a-f-c+h)} \exp[\sigma(K_1 s_a s_g + K_2 s_b s_f + K_3 s_d s_h + K_4 s_c s_e)] \tag{4.1}$$

where $s_a=(-1)^a$, $s_b=(-1)^b$, etc., are the spin variables used in ref. 5, with values ± 1. The coefficients $K_1,..., K_4$ are independent of the spins. They are functions of the three dihedral angles θ_1, θ_2, θ_3 which parametrize the Boltzmann weights of the model. A geometric interpretation of these angles is explained below in this section.

As in ref. 5, it is convenient to consider θ_1, θ_2, θ_3 as angles of a spherical triangle and define the spherical excesses $\alpha_0,..., \alpha_3$ by[6]

$$2\alpha_0=\theta_1+\theta_2+\theta_3-\pi, \qquad \alpha_i=\theta_i-\alpha_0 \tag{4.2}$$

for $i=1, 2, 3$. Choose θ_1, θ_2, θ_3 so that θ_1, θ_2, θ_3, $\alpha_0,..., \alpha_3$ are all real, between 0 and π. Further define

$$\begin{aligned} t_i &= [\tan(\alpha_i/2)]^{1/2}, \qquad i=0,..., 3 \\ T_i &= [\tan(\theta_i/2)]^{1/2}, \qquad i=1, 2, 3 \end{aligned} \tag{4.3}$$

[6] The reader should not confuse these notations with the notations for the spin variables used before.

and

$$\begin{aligned} x &= \operatorname{arctanh}(t_0/t_3), \qquad & x' &= \operatorname{arctanh}(t_1 t_2) \\ y &= \arctan(t_2/t_1), \qquad & y' &= \arctan(t_0/t_3) \end{aligned} \tag{4.4}$$

(choosing $x,..., y'$ to be real and nonnegative, y and y' to be not greater than $\pi/2$); then

$$\begin{aligned} 2K_1 &= -x' - iy', \qquad & 2K_2 &= x - iy' \\ 2K_3 &= -x' + iy', \qquad & 2K_4 &= x + iy' \end{aligned} \tag{4.5}$$

There are more useful expressions for $K_1,..., K_4$, found in ref. 5 [Eqs. (7.19) therein]. To write them down, set

$$v_i = \tanh 2K_i, \qquad i = 1,..., 4 \tag{4.6}$$

and let a_1, a_2, a_3 be the three sides of the spherical triangle, opposite to the angles θ_1, θ_2, θ_3. Then

$$\begin{aligned} v_1 &= -zT_1T_2, \qquad & v_2 &= -izT_2/T_1 \\ v_3 &= -z^{-1}T_1T_2, \qquad & v_4 &= iz^{-1}T_2/T_1 \end{aligned} \tag{4.7}$$

where $z = \exp(ia_3/2)$. Thus, $v_1,..., v_4$ satisfy the simple relation

$$v_1 v_4 + v_2 v_3 = 0 \tag{4.8}$$

One can conveniently parametrize them by introducing four new variables p, p', q, q':

$$v_1 = q'/p', \qquad v_2 = q'/p, \qquad v_3 = p/q, \qquad v_4 = -p'/q \tag{4.9}$$

Thus the ratios of p, p', q, q' are functions of θ_1, θ_2, θ_3. Conversely, one can express θ_1, θ_2, θ_3 through these ratios. From (4.7) and (4.9) one easily obtains

$$\begin{gathered} \tan\frac{\theta_1}{2} = i\frac{p}{p'}, \qquad \tan\frac{\theta_2}{2} = i\frac{q'}{q} \\ \exp(ia_3) = \frac{qq'}{pp'} \end{gathered} \tag{4.10}$$

Using these relations and the basic relation between the elements of the spherical triangle

$$\cos\theta_3 = \sin\theta_1 \sin\theta_2 \cos a_3 - \cos\theta_1 \cos\theta_2 \tag{4.11}$$

and performing some elementary calculations, one obtains

$$\tan\frac{\theta_3}{2}=i\left\{\frac{(p^2-q^2)(p'^2-q'^2)}{(p^2-q'^2)(p'^2-q^2)}\right\}^{1/2} \tag{4.12}$$

Next, from (4.6) and (4.9) it follows that

$$\begin{aligned} e^{-2K_1}&=(\Phi(1)\,w(p'/q',1))/w(p'/q',0)\\ e^{-2K_2}&=w(q'/p,1)/w(q'/p,0)\\ e^{-2K_3}&=w(p/q,1)/w(p/q,0)\\ e^{-2K_4}&=w(p'/q,0)/w(p'/q,1) \end{aligned} \tag{4.13}$$

where $w(x, l)$ and $\Phi(l)$ are defined by the relations (2.1)–(2.6) with $N=2$. Further, substituting these expressions into (4.1), one can easily check that up to a trivial scalar factor [which can be set to unity with the proper choice of $w(x, 0)$] the summand in (4.1) exactly coincides with the expression in the curly brackets in (2.10) for the case of $N=2$. Hence the weight function (2.10) for this case is essentially the same as that of (4.1), differing only by face-type equivalence factors unaffecting the partition function and by an overall normalization factor.

For the moment let us restrict ourselves to the layer homogeneous case and fix $N=2$. We have just shown that the model of Section 2 in this case reduces to the Zamolodchikov model. Hence, the modified model of Section 2 [which in the homogeneous n-layer case was shown to be equivalent to the critical $sl(n)$-chiral Potts model] is exactly the modified n-layer Zamolodchikov model considered in ref. 5. In that paper the partition functions per site in the thermodynamic limit were calculated for both of these models. Note, in particular, that for the first model these calculations are based on factorization properties of the transfer matrix and the symmetry properties of the model. The former follow directly from the expression (4.1) for the Boltzmann weights, while the latter are quite nontrivial for the representation (4.1): if Z denotes the partition function (1.1) for the M-site lattice, then $(\sin\theta_3)^{M/2} Z$ is unchanged[5] upon an arbitrary permutation of the variables $\alpha_0,\ldots,\alpha_3$ defined in (4.2). To prove this, one needs to restore all omitted equivalence transformation factors, returning to the original Zamolodchikov expressions for the Boltzmann weights of the model, which explicitly display this symmetry.

Another aspect of the rotational symmetry in the Zamolodchikov model we would like to mention is as follows. Suppose one considers this model on a finite lattice of size $l\times m\times n$. Then modulo the modification of the boundary condition (which constitutes the transfer to the modified model discussed in Section 2) it can be regarded either as the $sl(n)$-chiral

Potts model on a two-dimensional lattice of size $l \times m$ or, say, as the $sl(m)$-chiral Potts model on a lattice of size $l \times n$. Thus, the rank of the underlying algebra for the two-dimensional model is interpreted as a third dimension of the lattice, which seems to be quite unusual and interesting.

Note finally that the above method of the calculation of the partition function used in ref. 5 can be directly adopted to the case when $N > 2$, provided one could exhibit the rotational symmetry properties of the model generalizing, for example, the Zamolodchikov angle parametrization in this case. This problem is currently under investigation.

Let us now discuss the inhomogeneous Zamolodchikov model.[3] To describe it, first replace the cubic lattice $\mathscr{L}$ with its dual $\mathscr{L}_D$. The cube shown in Fig. 1 is then replaced by the three planes intersecting at a point (which is the site of $\mathscr{L}_D$), dividing the three-dimensional space into eight volumes with the spins $a,\dots, h$ assigned to them. Now $V(a,\dots, h)$ is regarded as the Boltzmann weight of the spin configuration of these eight volumes surrounding the site of $\mathscr{L}_D$. Let us interpret θ_1, θ_2, θ_3 as dihedral angles between the planes, arranging them as follows. Consider a sphere with the center at the site of $\mathscr{L}_D$. The above planes draw three great circles on the sphere dividing it into eight spherical triangles; θ_1, θ_2, θ_3 are exactly the intersection angles of these circles. The spins $a,\dots, h$ can now be associated with the interiors of these triangles. Performing the stereographic projection, one can map the sphere on the plane as shown in Fig. 12.

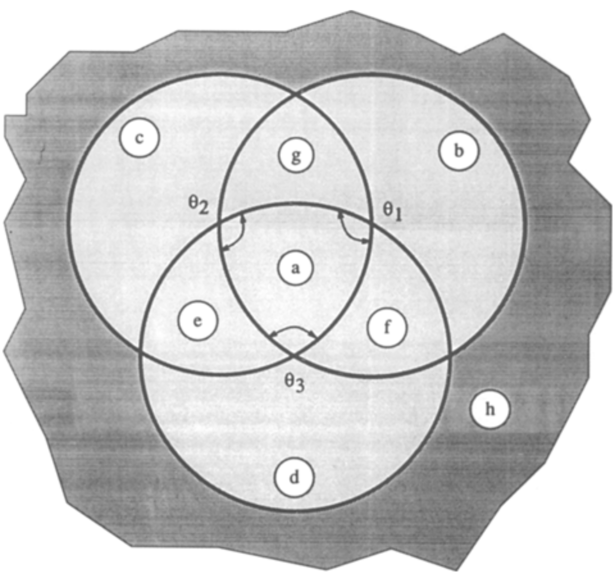


Fig. 12. Stereographic projection of the sphere surrounding the site of $\mathscr{L}_D$. The corner spins $a,\dots, h$ in Fig. 1 are now associated with interior areas of eight spherical triangles.

A nice feature of this geometric picture is that now one can define a "Z-invariant" solvable model[3,24] on the irregular three-dimensional lattice formed by an arbitrary intersection of planes, no four of which intersect at the same point. In such a model the Boltzmann weights associated with each intersection point of three planes depend on the dihedral angles between the planes at that point. Of course, for a regular lattice $\mathscr{L}_D$ with the coordination structure of the simple cubic lattice we get the same model as before. It will be formed by $l+m+n$ planes if we take its size as $l\times m\times n$. Consider now a small deformation $\mathscr{L}'_D$ of $\mathscr{L}_D$ preserving its coordination structure. Then its dual $\mathscr{L}'$ will have the same coordination structure and we can consider the model as the interaction-round-a-cube model, but the angles θ_1, θ_2, θ_3 which parametrize the Boltzmann weights will vary from one cube of the lattice to the other. Of course they do not vary arbitrarily: θ_1 (θ_2) is the same for any vertical (horizontal) line of adjacent cubes, while θ_3 is the same for any front-to-back line. Altogether one needs to specify

$$\#(\text{angles}) = 2(l+m+n)-3 \tag{4.14}$$

dihedral angles to fix the relative orientation of $l+m+n$ planes (three angles for the first three planes plus two angles for each additional plane).

Consider, for example, the parallelepiped $\mathscr{P}$ consisting of n cubes, shown in Fig. 2. The angles θ_1, θ_2 could vary from cube to cube while θ_3 remains the same for all the cubes in $\mathscr{P}$. Let us generalize (4.10), (4.12), introducing the corresponding variables $\theta_1^{(j)}$, $\theta_2^{(j)}$, p_j, p'_j, q_j, q'_j for the jth cube in $\mathscr{P}$. Then

$$\begin{gathered}\tan\frac{\theta_1^{(j)}}{2} = i\frac{p_j}{p'_j}, \qquad \tan\frac{\theta_2^{(j)}}{2} = i\frac{q'_j}{q_j}, \qquad \exp(ia_3^{(j)}) = \frac{q_j q'_j}{p_j p'_j} \\ \tan\frac{\theta_3}{2} = i\left\{\frac{(p_j^2-q_j^2)(p_j'^2-q_j'^2)}{(p_j^2-q_j'^2)(p_j'^2-q_j^2)}\right\}^{1/2}\end{gathered} \tag{4.15}$$

for any $j=1,\ldots,n$, where $a_1^{(j)}$, $a_2^{(j)}$, $a_3^{(j)}$ are the sides of the spherical triangle opposite to the angles $\theta_1^{(j)}$, $\theta_2^{(j)}$, θ_3. Comparing now the last of these relations with (2.12) (for $N=2$), one finds that we have exactly reproduced the parameter structure of the parallelepiped weight function S of the inhomogeneous model of Section 2. In fact, we can do this consistently for all front-to-back lines of cubes and reproduce the fully inhomogeneous model discussed at the end of Section 2, expressing $2n-3$ independent moduli of the curve (2.14) and $2(l+m)$ rapidity variables through $2(l+m+n)-3$ dihedral angles (the number of the parameters defining the model, of course, remains the same). These calculations are given below.

Consider the curve (3.1) for $N=2$ and introduce the set of n^2 two-by-two matrices $\{\mathbf{K}_{ij}\}$, $i, j=1,..., n$, defined by the relations

$$\begin{aligned}
&\mathbf{K}_{ij}=\mathbf{K}^{(i-1)}\mathbf{K}^{(i-2)}\cdots\mathbf{K}^{(j)}, \qquad i>j \\
&\mathbf{K}_{ii}=1 \\
&\mathbf{K}_{ij}=\mathbf{K}_{ij}^{-1}, \qquad i<j
\end{aligned} \tag{4.16}$$

where $\mathbf{K}^{(i)}$, $i=1,..., n-1$, are the same as in (3.1). Then,

$$\mathbf{K}_{ij}\mathbf{K}_{jk}\mathbf{K}_{ki}=1, \qquad \forall i, j, k \tag{4.17}$$

and, with account of (2.15),

$$\det \mathbf{K}_{ij}=1 \tag{4.18}$$

Further, denote

$$\begin{aligned}
\mathbf{L}_i&=\begin{pmatrix}(h_i^+(P))^2 & (h_i^+(P'))^2\\ (h_i^-(P))^2 & (h_i^-(P'))^2\end{pmatrix}\\
\mathbf{M}_i&=\begin{pmatrix}(h_i^+(Q))^2 & (h_i^+(Q'))^2\\ (h_i^-(Q))^2 & (h_i^-(Q'))^2\end{pmatrix}
\end{aligned} \tag{4.19}$$

Then from (3.1) it follows that

$$\mathbf{L}_i=\mathbf{K}_{ij}\mathbf{L}_j, \qquad \mathbf{M}_i=\mathbf{K}_{ij}\mathbf{M}_j, \qquad \forall i, j \tag{4.20}$$

Equations (4.18) and (4.20) ensure that both $\det \mathbf{L}_i$ and $\det \mathbf{M}_i$ are independent of i, so one can set

$$\det \mathbf{L}_i=\det \mathbf{M}_i=1, \qquad \forall i \tag{4.21}$$

by a trivial rescaling of h's. One could easily count that the matrices $\mathbf{K}_{ij}$, $\mathbf{M}_i$, $\mathbf{L}_i$, $i, j=1,..., n$, form a $(3n+3)$-parameter set. These parameters can conveniently be specified by the following construction. Consider the lattice $\mathscr{P}_D$ dual of the parallelepiped $\mathscr{P}$. It is formed by $n+2$ planes (horizontal, vertical, and n front-to-back planes). Let S_h, S_v, S_1,..., S_n denote the great circles corresponding to these planes as explained above. A typical part of the picture representing these circles is shown in Fig. 13 in the stereographic projection. Choose now arbitrary points A_h, A_v, A_1,..., A_n on each of these circles. Note that, up to overall rotations of the 3D space, such a configuration is defined by $3n+3$ parameters: $2n+1$ independent angles between the circles S_h, S_v, S_1, ..., S_n and $n+2$ arc lengths giving positions of the points A_h, A_v, A_1,..., A_n. Let r_h denote the arc OA_h, r_v

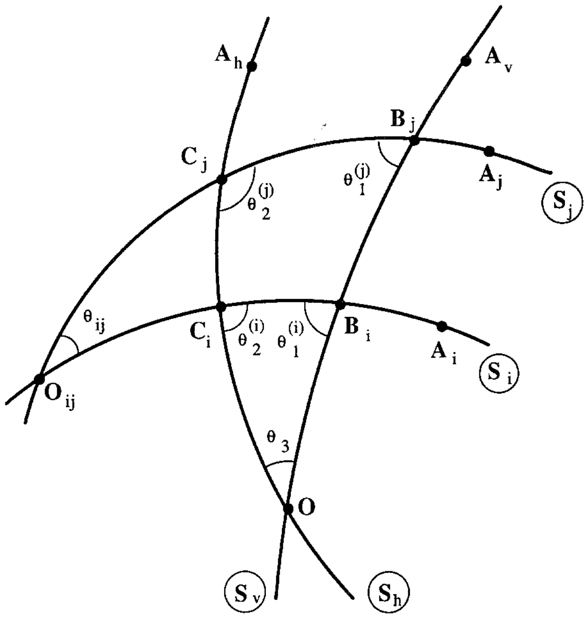


Fig. 13. A typical part of the picture representing the great circles corresponding to the planes which form the lattice $\mathscr{P}_D$ dual to the parallelepiped $\mathscr{P}$.

denote the arc OA_v, and r_i denote the arc A_iC_i, while θ_{ij} denotes the angle between S_i and S_j for $i, j = 1,..., n$. Comparing the Fig. 12 and Fig. 13, we can identify the various angles and arcs used in Eqs. (4.15) with the other elements in Fig. 13. Note, in particular, that $a_1^{(j)}$, $a_2^{(j)}$, $a_3^{(j)}$ in those equations are, respectively, the arcs OC_j, OB_j, B_jC_j in Fig. 13.

Below we need to use the following elementary formula, which can be found, e.g., in Eqs. (1)–(3″) on p. 100 in ref. 25 (provided one identifies the variables used therein with the elements of the spherical triangle).

Formula. Let θ_1, θ_2, θ_3 and a_1, a_2, a_3 be the angles and opposite sides of a spherical triangle, respectively. Then

$$\mathbf{U}(\theta_1)\,\mathbf{D}(\varepsilon a_3)\,\mathbf{U}(\theta_2) = \mathbf{D}(\varepsilon a_2)\,\mathbf{U}(\pi - \theta_3)\,\mathbf{D}(\varepsilon a_1), \qquad \varepsilon = \pm 1 \tag{4.22}$$

where **U** and **D** denote unitary matrices

$$\mathbf{U}(\theta) = \exp(i\theta\tau_1/2), \qquad \mathbf{D}(\phi) = \exp(i\phi\tau_3/2) \tag{4.23}$$

with τ_1, τ_2, τ_3 being the usual Pauli matrices.

Now set

$$\begin{aligned} \mathbf{L}_i &= \mathbf{D}(a_3^{(i)} - r_i)\,\mathbf{U}(\theta_1^{(i)})\,\mathbf{D}(-a_2^{(i)} + r_h) \\ \mathbf{M}_i &= \mathbf{D}(-r_i)\,\mathbf{U}(\pi - \theta_2^{(i)})\,\mathbf{D}(-a_2^{(i)} + r_v) \end{aligned} \tag{4.24}$$

Using (3.2) and similar expressions for p'_i, q_i, q'_i, one can easily check that (4.19) and (4.24) are consistent with (4.15). Expressing now $\mathbf{K}_{ij}$ from any of two relations (4.20) and applying the formula (4.22), we obtain

$$\mathbf{K}_{ij} = \mathbf{D}(-b_{ij} - r_i)\, \mathbf{U}(\theta_{ij})\, \mathbf{D}(b_{ji} + r_j) \tag{4.25}$$

where b_{ij} and b_{ji} denote, respectively, the arcs $O_{ij}C_i$ and $O_{ij}C_j$ in Fig. 13. The formulas (4.24), (4.25) give the required parametrization of Eqs. (3.1). Note that all the arcs $r_h, r_v, r_1,..., r_n$ which define the positions of the arbitrary points $A_h, A_v, A_1,..., A_n$ on the circles cancel out of the Boltzmann weight functions (2.10) and (3.7). All the other parameters entering (2.24), (4.25) can be expressed through the $2n+1$ independent angles which define the relative orientation of the $n+2$ planes forming $\mathscr{P}_D$. Further, for any two-by-two matrix $\mathbf{K}$ define a function

$$\xi(\mathbf{K}) = k_{12}k_{21}/k_{11}k_{22} \tag{4.26}$$

where k_{ab}, a, $b = 1, 2$, denote matrix elements of $\mathbf{K}$. Then, from (4.25),

$$\xi_{ij} = \xi(\mathbf{K}_{ij}) = -\tan^2 \frac{\theta_{ij}}{2}, \qquad i, j = 1,..., n \tag{4.27}$$

Obviously ξ_{ij} is the only interesting combination of the matrix elements of $\mathbf{K}_{ij}$ which is not affected by the transformations (2.16), (4.16).

5. THE TWO-LAYER CASE

In ref. 7 it is shown that the homogeneous two-layer Zamolodchikov model ($N = 2$) reduces to the critical two-dimensional free-fermion model, which is equivalent to the critical checkerboard Ising model.[8] Here we generalize this statement for the layer inhomogeneous model with arbitrary value of N.

Indeed it has been already shown in Section 3 that the modified n-layer three-dimensional model of Section 2 is equivalent to the $sl(n)$-CP model of ref. 17. When $n = 2$ (which is the two-layer case for the three-dimensional model) both models reduce to the (off-critical) chiral Potts model of ref. 11. The corresponding formulas are given below.

Consider the curve (3.1) for $n = 2$. In this case there is only one (matrix) equation in (3.1) involving only one moduli matrix $\mathbf{K}^{(1)}$. Applying (2.16), we can transform it to the form

$$\mathbf{K}^{(1)} = \frac{1}{k'} \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix} \tag{5.1}$$

where $k^2 + k'^2 = 1$ and k is an arbitrary parameter. From (2.14) it is clear that this parameter measures the inhomogeneity among the two layers of the three-dimensional model. When $k = 0$ the layers become identical.

Let us denote[7]

$$a_p = \omega^{-1/2} h_2^-(P), \qquad b_p = h_1^-(P), \qquad c_p = h_1^+(P), \qquad d_p = h_2^+(P) \tag{5.2}$$

With these notations (3.1) gives

$$\begin{aligned} a_p^N + k' b_p^N &= k c_p^N, & k' a_p^N + b_p^N &= k c_p^N \\ k a_p^N + k' c_p^N &= d_p^N, & k b_p^N + k' d_p^N &= c_p^N \end{aligned} \tag{5.3}$$

where each pair of (5.3) implies the other two. This is precisely the rapidity curve (9) of ref. 11. Let us write down the specialization of the curve (2.14) to $n = 2$. Introducing the variables $x_p = a_p/d_p$, $y_p = b_p/c_p$ and taking into account Eqs. (3.2), (5.2), one gets

$$x_p^N + y_p^N = k(1 + x_p^N y_p^N) \tag{5.4}$$

Adding now the superscript (CP) to the Boltzmann weights given in Eqs. (2), (3) of ref. 11, we can write them as

$$\frac{\bar{W}_{pq}^{(\mathrm{CP})}(m)}{\bar{W}_{pq}^{(\mathrm{CP})}(0)} = \prod_{j=1}^{m} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} \tag{5.5}$$

$$\frac{W_{pq}^{(\mathrm{CP})}(m)}{W_{pq}^{(\mathrm{CP})}(0)} = \prod_{j=1}^{m} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j} \tag{5.6}$$

where m is an integer. These Boltzmann weights satisfy the following symmetry relations [Eqs. (18a), (18b) of ref. 11]:

$$\begin{gathered} \bar{W}_{pq}^{(\mathrm{CP})}(m) = W_{qRp}^{(\mathrm{CP})}(m), \qquad W_{pq}^{(\mathrm{CP})}(-m) = W_{RpRq}^{(\mathrm{CP})}(m) \\ W_{pq}^{(\mathrm{CP})}(m)\, W_{qp}^{(\mathrm{CP})}(m) = W_{pq}^{(\mathrm{CP})}(0)\, W_{qp}^{(\mathrm{CP})}(0) \end{gathered} \tag{5.7}$$

where R denotes an automorphism of the curve (5.2) defined as

$$(a_{Rp}, b_{Rp}, c_{Rp}, d_{Rp}) = (b_p, \omega a_p, d_p, c_p) \tag{5.8}$$

[7] Note that P and p here have the same meaning: they refer to the same point on the curve (3.1). Below in this section we will use both upper case and lower case letters for the same rapidity variable (like P and p above) to match both the notation used in the preceding sections and the notation used in ref. 11. We hope that this will not cause any confusion for the reader.

For $n=2$ the Boltzmann weight function (3.7) depends on only two spin differences $\hat{\alpha}_1$ and $\hat{\beta}_1$ defined in (3.4). Denoting them simply as $\hat{\alpha}$ and $\hat{\beta}$, respectively, and substituting (5.2) into (3.12), one gets

$$\frac{\bar{W}_{PQ}(\alpha, \beta)}{\bar{W}_{PQ}(0, 0)} = \Phi^{-1}(\hat{\alpha})\, \Phi(\hat{\beta})\, \frac{\bar{W}^{(\mathrm{CP})}_{pq}(\hat{\alpha}-\hat{\beta})}{\bar{W}^{(\mathrm{CP})}_{pq}(0)} \tag{5.9}$$

where Φ is defined in (2.6). Further, the R-matrix (3.15) in the considered case is identical with the R-matrix of the checkerboard chiral Potts model given in ref. 11. Indeed, substituting (5.9) into (3.15) and using (5.7), we obtain (up to an inessential normalization factor)

$$\begin{aligned} &R(P, P', Q, Q' \mid \alpha, \beta, \gamma, \delta) \\ &\quad = \bar{W}^{(\mathrm{CP})}_{pq}(\hat{\alpha}-\hat{\delta})\, \bar{W}^{(\mathrm{CP})}_{Rp'Rq'}(\hat{\beta}-\hat{\gamma})\, W^{(\mathrm{CP})}_{pRq'}(\hat{\delta}-\hat{\gamma})\, W^{(\mathrm{CP})}_{Rp'q}(\hat{\alpha}-\hat{\beta}) \end{aligned} \tag{5.10}$$

which is exactly expression (20) of ref. 11 with their rapidities (p_1, p_2, q_1, q_2) replaced by our (p, Rp', q, Rq') and their spins $(\alpha, \mu, \beta, \lambda)$ replaced by our $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$, respectively.

Thus the have proved the required equivalence of the modified two-layer, three-dimensional model of the present paper with the checkerboard chiral Potts model of ref. 11. As was remarked before, the parameter k measures the layer inhomogeneity of the three-dimensional model. On the other hand, it is a temperature-like variable in the chiral Potts model. It is critical when $k=0$. The expression for the local state probabilities for the central spin α conjectured in ref. 26 reads

$$\langle \omega^{\alpha j} \rangle = (k^2)^{\Delta_j}, \qquad \Delta_j = \frac{j(N-j)}{2N^2}, \qquad j = 1, \ldots, N-1 \tag{5.11}$$

In particular, when $N=2$, (5.11) reduces to the Onsager and Yang expression[27,28] for the spontaneous magnetization of the Ising model, while (4.27) gives

$$k^2 = -\tan^2 \frac{\theta_{12}}{2} \tag{5.12}$$

where θ_{12} is the dihedral angle between the front-to-back planes of two layers of the three-dimensional model.

6. DISCUSSION AND CONJECTURES

Here we give several remarks outlining consequent problems under investigation, indicating some interesting connections of the model with other subjects.

6.1. The Rotational Symmetry

Is the Boltzmann weight function (2.9) symmetric with respect to spatial rotations of the cube in Fig. 1? If so, can one generalize Zamolodchikov's angle parametrization for $N>2$?

6.2. The Local Integrability Conditions

We believe that the Boltzmann weights (2.9) satisfy the following three-dimensional star–star relation:

$$\frac{\bar{V}(a\,|\,efg\,|\,bcd\,|\,h)}{V(a\,|\,efg\,|\,bcd\,|\,h)}=\frac{w(z,c-h-g+b)\,s(g+h,g-b)}{w(z,e-d-a+f)\,s(a+d,a-f)} \tag{6.1}$$

where $\bar{V}(a,..., h)$ is given by (2.9) with $v_\sigma(a,..., h)$ replaced by

$$\begin{aligned}\bar{v}_\sigma&(a\,|\,efg\,|\,bcd\,|\,h)\\ &=w_{q'q}(d-h-f+b)\,w_{q'q}^{-1}(e-c-a+g)\,s(c,g-a)\,s(h,f-b)\\ &\quad\times\{w_{p'q}^{-1}(\sigma-f+b)\,w_{pq}(\sigma-a+g)\,w_{q'p}(e-c-\sigma)\\ &\quad\times\tilde{w}_{p'q'}(\sigma-d+h)\,s(-\sigma,a-c-f+h)\}\end{aligned} \tag{6.2}$$

where

$$z=e^{-i\pi/N}(\Gamma(p,p',q,q'))^{1/N} \tag{6.3}$$

with Γ defined in (2.12a). The phases of the Nth roots are fixed as an analytic continuation from the case when $q=1$ and p, p', q' are positive real and $0<p'<q'<p<1$. In this case all the arguments of the Nth roots in (2.3) and (6.3) are real and positive and we take positive values of the roots. For $N=2$ the relation (6.1) is exactly the relation (A.2) of ref. 5. We have verified (6.1) numerically for $N=3$ and $N=4$, but at the moment it remains a conjecture, as we do not have a complete proof.

6.3. The Functional Equations

We hope that one can generalize the functional equations of refs. 15 and 29 to the case of the $sl(n)$-CP model. Note, in particular, that for the $n=3$, $N=2$ case the τ–τ equations (in the terminology of ref. 29) have a very simple form[30]

$$\begin{aligned}\tau(x)\,\tau(xq)&=\phi_1(x)+\bar{\tau}(x)+\bar{\tau}(xq)\\ \bar{\tau}(x)\,\bar{\tau}(xq)&=\phi_2(x)+\phi_3(x)\,\tau(xq)+\phi_4(x)\,\tau(x)\end{aligned} \tag{6.4}$$

where $q^2 = -1$, $\tau(x)$ is the transfer matrix constructed with the L-operator (2.2) of ref. 17 related to the vector representation of $sl(3)$, while $\bar{\tau}(x)$ is a similar transfer matrix related to an antisymmetric tensor representation, and $\phi_i(x)$, $i = 1,..., 4$, are known scalar functions.

6.4. The $sl(n)$-Parafermion Conformal Field Theory

It was argued in ref. 20 that the scaling limit in the critical $sl(n)$-CP model is described by $sl(n)$-parafermion conformal field theory.[21,22] If so, then this theory in the proper $n \to \infty$ limit could describe the scaling behavior of the three-dimensional model.

6.5. The Quantum Groups at "Roots of Unity"

The R-matrix (3.15) has been found[17] as the solution of the Yang–Baxter equation intertwining two cyclic L-operators related to the representation of the (modified) $U_q(sl(n))$ algebra with $q^{2N} = 1$. Alternatively, it can be described[23] as the intertwiner of two cyclic representations of the $U_q(\widehat{sl(n)})$ algebra. This raises the question of the meaning of the three-dimensional structure of the $sl(n)$-CP model for the cyclic representations of these algebras.

6.6. Are There Other Two-Dimensional Solvable Models Which Admit a Three-Dimensional Interpretation?

There are two known off-critical deformations of the Fateev–Zamolodchikov Z_N model.[13] The first is the original ($n = 2$) chiral Potts model,[11] the second is the Kashiwara–Miwa "broken Z_N" model.[31] Note that the latter can be regarded as a "descendant of the 8-vertex model"[32] through the generalization of the algebraic construction of ref. 15. As remarked in ref. 17, one can apparently generalize these calculations starting with Belavin's elliptic R-matrix[33] that should result in the "$sl(n)$-generalized broken Z_N model." This model should intersect with the $sl(n)$-CP model at criticality and therefore could admit the three-dimensional interpretation as well.

6.7. The Magnetic Monopoles in Three Dimensions

There is the result by Atiyah and Murray reported in ref. 34 that the rapidity curve (5.4) of the ($n = 2$) chiral Potts model exactly coincides with the spectral curve which determines a special $SU(2)$ Bogomol'ny

N-monopole with cyclic Z_N-symmetry in the hyperbolic 3-space. Murray informed us[35] that this connection can be extended when the gauge group is $SU(n)$ relating the curve (2.14) with the spectral curve determining an $SU(n)$ monopole solution. Unfortunately, these connections do not go beyond the coincidence of the algebraic curves: there is still no interpretation of the Yang–Baxter equation in the monopole theory.

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REFERENCES

1. L. D. Faddeev, *Sov. Sci. Rev. C* **1**:107–155 (1980).
2. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
3. A. B. Zamolodchikov, *Zh. Eksp. Teor. Fiz.* **79**:641–664 (1980) [*JETP* **52**:325–336 (1980)]; A. B. Zamolodchikov, *Commun. Math. Phys.* **79**:489–505 (1981).
4. R. J. Baxter, *Commun. Math. Phys.* **88**:185 (1983).
5. R. J. Baxter, *Physica* **18D**:321–347 (1986).
6. V. V. Bazhanov and Yu. G. Stroganov, *Teor. Mat. Fiz.* **63**:417 (1985) [*Theor. Math. Phys.* **63**:604 (1985)].
7. R. J. Baxter and P. J. Forrester, *J. Phys. A* **18**:1483 (1985).
8. R. J. Baxter, *Proc. Roy. Soc. A* **404**:1–83 (1986).
9. H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M. Yan, *Phys. Lett. A* **123**:219 (1987).
10. B. M. McCoy, J. H. H. Perk, S. Tang, and C. H. Sah, *Phys. Lett. A* **125**:9 (1987).
11. R. J. Baxter, J. H. H. Perk, and H. Au-Yang, *Phys. Lett. A* **128**:138 (1988).
12. A. B. Zamolodchikov and V. A. Fateev, *Sov. Phys. JETP* **62**:215 (1985).
13. V. A. Fateev and A. B. Zamolodchikov, *Phys. Lett.* **92A**:37 (1982).
14. R. J. Baxter, Hyperelliptic function parametrization for the chiral Potts model, in *Proceedings of the International Congress of Mathematicans, Kyoto*, 1990 (Springer-Verlag, 1991), pp. 1305–1317.
15. V. V. Bazhanov and Yu. G. Stroganov, *J. Stat. Phys.* **59**:799 (1990).
16. I. V. Cherednik, *Teor. Mat. Fiz.* **43**(1):117–119 (1980);
P. P. Kulish and E. K. Sklyanin, in *Zapiski Nauch. Semin. LOMI*, Vol. 95, pp. 129–160;
J. H. H. Perk and C. L. Schultz, *Phys. Lett. A* **84**:407 (1981).
17. V. V. Bazhanov, R. M. Kashaev, V. V. Mangazeev, and Yu. G. Stroganov, *Commun. Math. Phys.* **138**:393–408 (1991).
18. V. V. Bazhanov and Yu. G. Stroganov, *Teor. Mat. Fiz.* **52**:105–113 (1982) [*Theor. Math. Phys.* **52**:685–691 (1982)].
19. M. T. Jaekel and J. M. Maillard, *J. Phys. A* **15**:1309 (1982).
20. R. M. Kashaev, V. V. Mangazeev, and T. Nakanishi, *Nucl. Phys. B* **362**:563 (1991).

21. M. Ninomiya and K. Yamagishi, *Phys. Lett. B* **183**:323–330 (1987).
22. D. Gepner, *Nucl. Phys. B* **290**[FS20]:10 (1987).
23. E. Date, M. Jimbo, K. Miki, and T. Miwa, *Commun. Math. Phys.* **137**:133 (1991).
24. R. J. Baxter, *Phil. Trans. R. Soc.* **289**:315–346 (1978).
25. N. J. Vilenkin, *Special Functions and the Theory of Group Representations* (American Mathematical Society, Providence, Rhode Island, 1968).
26. G. Albertini, B. M. McCoy, J. H. H. Perk, and S. Tang, *Nucl. Phys. B* **314**:741 (1989); H. Au-Yang and J. H. H. Perk, Onsager's star–triangle equation: Master key to integrability, in *Advanced Studies in Pure Mathematics* (Academic Press, 1989), Vol. 19, pp. 57–94.
27. L. Onsager, *Nuovo Cimento* (*Suppl.*) **6**:261 (1949).
28. C. N. Yang, *Phys. Rev.* **85**:808 (1952).
29. R. J. Baxter, V. V. Bazhanov, and J. H. H. Perk, *Int. J. Mod. Phys. B* **4**:803–870 (1990).
30. V. V. Bazhanov and R. M. Kashaev, unpublished (1990).
31. M. Kashiwara and T. Miwa, *Nucl. Phys. B* **275**[FS17]:121–134 (1986).
32. K. Hasewaga and Y. Yamada, *Phys. Lett. A* **146**:387–396 (1990).
33. A. A. Belavin, *Nucl. Phys. B* **180**[FS2]:189 (1981).
34. M. F. Atiyah, *Int. J. Mod. Phys. A* **6**:2761–2774 (1991).
35. M. K. Murray, Private communication (1991).